

# A NOTE ON TWO CONJECTURES ON DIMENSION FUNCTIONS OF $C^*$ -ALGEBRAS.

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**ABSTRACT.** Let  $A$  be an arbitrary  $C^*$  algebra. In [3] Blackadar and Handelman conjectured the set of lower semicontinuous dimension functions on  $A$  to be pointwise dense in the set  $DF(A)$  of all dimension functions on  $A$  and  $DF(A)$  to be a Choquet simplex. We provide an equivalent condition for the first conjecture for unital  $A$ . Then by applying this condition we confirm the first Conjecture for all unital  $A$  for which either the radius of comparison is finite or the semigroup  $W(A)$  is almost unperforated. As far as we know the most general results on the first Conjecture up to now assumes exactness, simplicity and moreover stronger regularity properties such as strict comparison. Our results are achieved through applications of the techniques developed in [4] and [13].

We also note that, whenever the first Conjecture holds for some unital  $A$  and extreme boundary of the the quasitrace simplex of  $A$  is finite, then every dimension function of  $A$  is lower semicontinuous and  $DF(A)$  is affinely homeomorphic to the quasitrace simplex of  $A$ . Combining this with the said results on the first Conjecture give us a class of algebras for which  $DF(A)$  is a Choquet simplex, i.e. gives a new class for which the 2nd Conjecture mentioned above holds.

## 1. INTRODUCTION.

In [8] (c.f [7] ) Cuntz introduced the subequivalence relation  $\preceq$  and used the relation to define dimension functions on (simple unital)  $C^*$ -algebras. Cuntz then associated the partially ordered abelian group  $K_0^*(A)$  to a (simple unital)  $C^*$ -algebra  $A$  and showed that dimension functions on  $A$  bijectively correspond to states on  $K_0^*(A)$ , making available the methods of [10] to study dimension functions on  $C^*$ -algebras. As a group,  $K_0^*(A)$  is the Grothendieck group of the Cuntz semigroup  $W(A)$  which is the natural extension of the Murray-von Neumann semigroup of projections  $V(A)$  to positive elements in matrix algebras over  $A$ .

Continuing from [8], Handelman [11] and later Blackadar and Handelman [3] developed a more general and a detailed theory for dimension functions on  $C^*$ -algebras. We focus on two Conjectures posted in [3];

**Conjecture 1.1.** [3] *For any  $C^*$ - algebra  $A$ , the set of lower semicontinuous dimension functions  $LDF(A)$  is dense in  $DF(A)$  in the topology of point wise convergence.*

**Conjecture 1.2.** [3] *The affine space  $DF(A)$  is a Choquet simplex for any  $C^*$ - algebra  $A$ .*

As shown in [3], there is an affine and bijective natural map from the set of normalized 2-quasitraces  $QT(A)$  of a  $C^*$ -algebra  $A$  to the set  $LDF(A)$ , whose inverse

is continuous. On the other hand Choquet simplexes are a natural extension of classical (finite) simplexes and the theory on Choquet simplexes is well developed - see [9]. Thus, the Conjectures (if true), provide useful tools that can be applied to derive properties of  $DF(A)$ .

For non stably finite  $C^*$ -algebras the Conjectures hold trivially, as in this case  $K_0^*(A) = 0$  and  $DF(A)$  is the empty set. In the stably finite case there are several classes for which at least one of the Conjectures are known to hold, as outlined below.

Conjecture 1.1 holds for unital commutative  $A$  by [3, Theorem I.2.4]. In [12, Corollary 4.4], Conjecture 1.2 is verified for unital (stably finite)  $C^*$ -algebras of real rank zero and stable rank one, providing the first (non trivial) examples for 1.2. The most general results on the Conjectures that we are aware of appear in [5]. Theorem B of [5] show that conclusions of both the Conjectures hold for unital, simple, separable, stably finite  $C^*$ -algebras which are either exact and  $\mathcal{Z}$ -stable or are  $AH$ -algebras of slow dimension growth. Furthermore [5, Remark 6.5] asserts that for exact  $A$  Conjecture 1.1 holds assuming strict comparison instead of  $\mathcal{Z}$ -stability. Applying the methods in [5], several classes of continuous fields of  $C^*$ -algebras for which 1.1 and 1.2 hold are provided in [1].

The above mentioned verifications have arisen more or less as applications of structure Theorems for  $W(A)$  (i.e. [12, Theorem 2.8] and [5, Theorems 6.4 and 6.6]). Apart from their usefulness in establishing the Conjectures these structure Theorems have other important applications - see [6] for an example. However, when concerning the conjectures alone such Theorems are too strong requirements to ask for, at least if the Conjectures are to hold in full generality.

To our knowledge there has not been any work focused on the conjectures alone and this paper is an attempt for a step in that direction. We aim to apply theory on state spaces of partially ordered semigroups developed mainly in [4] (c.f [10]) to study the conjectures. As it turns out this can be readily achieved, especially in the case of 1.1.

In particular these techniques (of [4]) allow us to prove the following Theorem which give an alternate form of Conjecture 1.1 for unital  $A$ .

**Theorem 3.1** Let  $A$  be a unital  $C^*$ -algebra. Then  $LDF(A)$  is dense in  $DF(A)$  if and only if  $\iota : W(A) \rightarrow LAff_b(QT(A))^+$  is a stable order embedding.

By  $LAff_b(QT(A))^+$  we mean the scaled partially ordered abelian semigroup of bounded non negative lower semicontinuous affine maps on  $QT(A)$ .  $\iota$  is the natural map given by  $\iota(\langle a \rangle)(\tau) = \lim_{n \rightarrow \infty} \tau(a^{1/n})$ ,  $\forall \tau \in QT(A)$ . In a sense this is a weaker form of the representation of  $W(A)$  given in [5, Theorem 6.4].

Using the above we prove;

**Theorem 3.3** Let  $A$  be any unital  $C^*$ -algebra. The following hold.

1. If  $A$  has finite radius of comparison then  $LDF(A)$  is dense in  $DF(A)$ .
2. If  $W(A)$  almost unperforated then  $LDF(A)$  is dense in  $DF(A)$ .
3. If  $\partial_e(QT(A))$  is a finite set and if either of the assumptions above (in 1,2) holds for  $A$  then  $DF(A) = LDF(A)$  and  $DF(A)$  is affinely homeomorphic to  $QT(A)$ . In particular  $DF(A)$  is a Choquet simplex.

To prove (1) and (2), we verify that the alternate form of Conjecture 1.1 provided in Theorem 3.1 hold in the respective classes. In (1) this is done by applying techniques of [4] once more while in the second case this is done by following the ideas of [13]. Combining the conclusions of parts 1 and 2 with Lemma 3.3 - which mainly is a consequence of Krein-Milman Theorem - we prove (3).

These results greatly extend the class of unital  $C^*$ -algebras for which the Conjectures (specially 1.1) were known to hold. On the one hand these results do not assume simplicity or exactness as in [5] and on the other hand finite radius of comparison is a considerably weaker assumption than any of the regularity assumptions considered in [5]. Most of the continuous fields considered in [1] are also known to have finite radius of comparison.

In particular, the counter examples for Elliott's classification conjecture constructed in [14] and Villadsen algebras of type I [15] have finite radius of comparison but are not covered by [5]. Furthermore Villadsen algebras of type II ([16]) are of finite radius comparison and have unique quasitrace, and thus satisfy both the Conjectures from Theorem 3.4. This means that for each  $n \in \mathbb{N}$ , we now know that there are unital algebras of stable rank  $n$  which satisfy the Conjectures 1.1 and 1.2.

For simple  $C^*$ -algebras, almost unperforation of  $W(A)$  is equivalent to strict comparison (i.e zero radius of comparison) and thus the second case may seem somewhat redundant when compared to 1. However, in general (without simplicity) it is not clear how the two properties relate to each other.

In the next section we recall some preliminary results and notations that we require. Section 3 contain the proofs of the main results.

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## 2. PRELIMINARIES AND NOTATIONS

**2.1. Partially ordered abelian semigroups.** All semigroups we consider will be abelian. In addition we assume all semigroups to contain the identity element 0.

**Definition 2.1.** A partially ordered semigroup is a pair  $(M, \leq)$  where  $M$  is a abelian semigroup and  $\leq$  is a partial order on  $M$  such that  $\forall a, b, c \in M, a \leq b \implies a + c \leq b + c$ . We also assume that  $0 \leq a$  for all  $a \in M$ .

*Remark.* The term partially ordered semigroup is used even without assuming  $0 \leq a, \forall a \in M$  and the term positively ordered semigroup is used for ones which in addition satisfy this. We do not have a need distinguish the two cases.

All order relations we consider will be partial orders and for convenience we write ordered semigroup to mean a partially ordered semigroup in the sense of 2.1.

An element  $u$  in  $(M, \leq)$  is called a order unit if for each  $x \in M$  there is some  $n \in \mathbb{N}$  with  $x \leq nu$ . A triple  $(M, \leq, u)$  where  $(M, \leq)$  and  $u$  are as above is called a scaled ordered semigroup. If the order and the order unit are clear we may write  $M$  to denote  $(M, \leq, u)$ .

A morphism from  $(M, \leq, u)$  to  $(N, \leq, v)$  is a map  $\phi : M \rightarrow N$  which is additive and order preserving with  $\phi(0) = 0$  and  $\phi(u) = v$ .

A state on  $(M, \leq, u)$  is a morphism from  $M$  to  $(\mathbb{R}^+, \leq, 1)$  where  $\mathbb{R}^+$  is the additive semigroup of non negative real numbers and  $\leq$  is as usual. The set of all states of  $(M, \leq, u)$  will be denoted by  $S(M, \leq, u)$  (or by  $S(M)$  if the choice of order unit and order are clear).  $S(M)$  is compact and convex as a subset of the space of all real valued functions on  $M$  in the topology of pointwise convergence.

A morphism  $\phi : (M, \leq, u) \rightarrow (N, \leq, v)$  as above induce a continuous affine map  $\phi^\# : S(N) \rightarrow S(M)$  via composition.

The following class of morphisms between scaled ordered semigroups was introduced in [4].

**Definition 2.2.** [4, Definition 2.2] Let  $(M, \leq, u), (N, \leq, v)$  be scaled ordered semigroups and  $\phi : M \rightarrow N$  be a morphism of scaled ordered semigroups.  $\phi$  is called a stable order embedding if for any  $x, y \in M$ , there are  $n \in \mathbb{N}$  and  $z \in M$  with  $nx + z + u \leq ny + z$  if and only if there are  $m \in \mathbb{N}$  and  $w \in N$  with  $m\phi(x) + v + w \leq m\phi(y) + w$ .

We recall some useful results from [4].

**Lemma 2.3.** [4, Lemma 2.8] *Let  $(M, \leq, u)$  be a scaled ordered semigroup and  $x, y \in M$ . Then  $s(x) < s(y)$  for all  $s \in S(M)$  if and only if there is some  $n \in \mathbb{N}$  and  $z \in M$  such that  $nx + z + u \leq ny + z$ .*

**Lemma 2.4.** [4, Lemma 2.9] *Let  $(M, \leq, u)$  be a scaled ordered semigroup and  $K$  be a nonempty compact convex subset  $S(M)$ . Suppose for any  $a, b \in M$  if  $s(a) < s(b)$  for all  $s \in K$  then  $s(a) < s(b)$  for all  $s \in S(M)$ . Then  $K = S(M)$ .*

**Theorem 2.5.** [4, Theorem 2.6] *Let  $\phi : (M, \leq, u) \rightarrow (N, \leq, v)$  be a morphism of scaled ordered semigroups. Then  $\phi$  is a stable order embedding iff*

$$S(M) = \{g \circ \phi : g \in S(N)\}.$$

*Remark 2.6.* From [4] the above statements hold even when  $M, N$  are pre-ordered. As we will only be considering partially ordered semigroups, we limit to this case.

### Almost unperforation.

**Definition 2.7.** An ordered semigroup  $(M, \leq)$  is said to be almost unperforated if  $kx \leq k'y \implies x \leq y$  for all  $x, y \in M$  and each  $k, k' \in \mathbb{N}$  with  $k > k'$

We will need the following Proposition which is proven in [13] (c.f. [2])

**Proposition 2.8.** [13, Proposition 3.2] *Let  $(M, \leq)$  be almost unperforated and  $u, x \in M$ . If  $u$  is a order unit and  $s(x) < s(u)$  for all  $s \in S(M, \leq, u)$  then  $x < u$ .*

**2.2. The Cuntz semigroup of a  $C^*$ -algebra and the group  $K_0^*(A)$ .** As usual, let  $W(A)$  denote the Cuntz semigroup of the  $C^*$ -algebra  $A$ .

That is,

$$W(A) = M_\infty(A)_+ / \sim$$

where  $\sim$  is the Cuntz equivalence relation. Recall that for  $a, b \in M_\infty(A)_+$ ,  $a$  is said to be Cuntz equivalent to  $b$  (written  $a \sim b$ ) iff  $a \preceq b$  and  $b \preceq a$ .  $W(A)$  inherits the natural partial order given by

$$\langle a \rangle \leq \langle b \rangle \iff a \preceq b,$$

and the pair  $(W(A), \leq)$  form a ordered semigroup. If  $A$  is unital then  $\langle 1_A \rangle$  is an order unit for  $(W(A), \leq)$  and will always be the chosen order unit of  $W(A)$  for us.

Following the notation of [8] let us write  $K_0^*(A)$  to denote the Grothendieck group of  $W(A)$  and set

$$K_0^*(A)_{++} = \{\gamma(y) - \gamma(x) : x, y \in W(A) \text{ and } x \leq y\}$$

where  $\gamma : W(A) \rightarrow K_0^*(A)$  denotes the natural map given by Grothendieck construction.

From [3] (c.f [8]) the pair  $(K_0^*(A), K_0^*(A)_{++})$  form a partially ordered abelian group and  $\gamma(\langle 1_A \rangle)$  is a order unit for  $(K_0^*(A), K_0^*(A)_{++})$ . Note also that  $(K_0^*(A), K_0^*(A)_{++})$  is directed; i.e.  $K_0^*(A) = K_0^*(A)_{++} - K_0^*(A)_{++}$ .

A state  $s$  on  $(G, G_+, u)$  where  $(G, G_+)$  is a partially ordered abelian group and  $u$  is an order unit, is an additive map  $s : G \rightarrow \mathbb{R}$  satisfying  $s(G_+) \subset [0, \infty)$  and  $s(u) = 1$ . Set of all states on  $(G, G_+, u)$  is denoted by  $S(G, G_+, u)$  or just by  $S(G)$  when there is no room for confusion. As in the semigroup case  $S(G, G_+, u)$  is a compact convex space. For detailed discussion on partially ordered abelian groups and their states see [9].

### 2.3. Dimension functions, Lower semicontinuous dimension functions and Quasitraces of a $C^*$ -algebra.

**Definition 2.9.** ([3, Definition I.1.2] c.f [8]) A dimension function on a unital  $C^*$ -algebra  $A$  is a function  $d : M_\infty(A)_+ \rightarrow [0, \infty)$  which satisfies the following conditions;

1.  $d(1_A) = 1$ .
2.  $d(a + b) = d(a) + d(b)$  for all  $a, b \in M_\infty(A)_+$  with  $a \perp b$ .
3.  $d(a) \leq d(b)$  for all  $a, b \in M_\infty(A)_+$  with  $a \preceq b$ .

*Remark.* In [3] dimension functions are defined on all elements in  $M_\infty(A)$  with some additional requirements. Its easily seen that the two definitions are equivalent and by replacing (1) above with the condition  $\sup\{d(a) : a \in A_+\} = 1$  the definition extends to non unital algebras.

The set of all dimension functions on  $A$  is denoted by  $DF(A)$ .

**Lemma 2.10.** ([3] c.f [8]) For a unital  $C^*$ -algebra  $A$ ,  $DF(A)$  is in bijective correspondence with  $S(K_0^*(A), K_0^*(A)_{++}, \gamma(\langle 1_A \rangle))$  and thus with  $S = S(W(A), \leq, \langle 1_A \rangle)$ .

*Proof.* We outline the identifications involved.

Any  $s \in S(W(A))$  uniquely determines a state  $s'$  on  $K_0^*(A)$  which is given by  $s'(\gamma(x) - \gamma(y)) = s(x) - s(y)$ . Conversely if  $s' \in S(K_0^*(A))$  then  $s(x) = s'(\gamma(x))$  is a state on  $W(A)$ . The map  $s \mapsto s'$  sets up a natural affine homeomorphism between  $S(W(A))$  and  $S(K_0^*(A))$ .

On the other hand if  $f \in DF(A)$  then  $s_f(\langle a \rangle) = f(a)$  is a state on  $W(A)$  and if  $s \in S(W(A))$  then  $f_s$  defined on  $M_\infty(A)_+$  by  $f_s(a) = s(\langle a \rangle)$  is in  $DF(A)$ .  $\square$

We will use the above identification freely.

Given  $a \in M_\infty(A)_+$  and  $\epsilon > 0$ , by  $(a - \epsilon)_+$  we denote the element of  $C^*(a)$  which corresponds (via the functional calculus of  $a$ ) to the function

$$f_\epsilon(t) = \max\{t - \epsilon, 0\}, t \in \sigma(a),$$

where  $\sigma(a)$  is the spectrum of  $a$ .

A dimension function  $s$  is said to be lower semicontinuous if for each  $a \in M_\infty(A)_+$

$$s(\langle a \rangle) \leq \liminf_n s_n(\langle a_n \rangle)$$

whenever  $(a_n)$  is a sequence in  $M_\infty(A)_+$  converging to  $a$  in norm. The above is equivalent to the requirement;

$$s(\langle a \rangle) = \sup_{\epsilon > 0} s(\langle (a - \epsilon)_+ \rangle)$$

for all  $a \in M_\infty(A)_+$ .

The set of all lower semicontinuous dimension functions of  $A$  is denoted by  $LDF(A)$ .

Recall that a quasitrace [3, Definition II.1.1] is a complex-valued function on a  $C^*$ -algebra having all the usual properties of a tracial state, but with linearity assumed only on commutative  $C^*$ -subalgebras. A 2-quasitrace is a quasitrace on  $A$  that extends to  $M_2(A)$ . From [3, Proposition II.4.1], any 2-quasitrace extends to  $M_n(A)$  for all  $n \in \mathbb{N}$ .

A quasitrace  $\tau$  is said to be normalized if  $\|\tau\| = \sup\{\tau(a) : a \in A_+, \|a\| \leq 1\} = 1$ . In the case that  $A$  is unital this is equivalent to  $\tau(1_A) = 1$ .

As usual  $QT(A)$  denotes the set of all normalized quasitraces of  $A$ .

Given  $\tau \in QT(A)$  define  $d_\tau : W(A) \rightarrow [0, \infty)$  by

$$d_\tau(\langle a \rangle) = \lim_{n \rightarrow \infty} (a^{1/n}).$$

**Theorem 2.11.** [3, Theorems II.2.2 and II.3.1] *Let  $A$  be a  $C^*$ -algebra. The map  $d_\tau$  is a lower semicontinuous dimension function on  $A$  for each  $\tau \in QT(A)$ . The assignment  $\tau \mapsto d_\tau$  gives an affine bijection from  $QT(A)$  onto  $LDF(A)$  which has a continuous inverse with respect to the pointwise topologies on both ends.*

For unital  $A$ ,  $QT(A)$  is a Choquet simplex [3, Theorem II.4.4]. Given a Choquet simplex  $K$ , the set of all non negative valued bounded lower semicontinuous affine maps from  $K$  into  $\mathbb{R}$  is denoted  $LAff_b(K)^+$ . With pointwise addition and pointwise ordering  $LAff_b(K)^+$  form a ordered semigroup - see [9] for a detailed discussion on these topics.

For  $\langle a \rangle \in W(A)$  define  $\iota(\langle a \rangle) : QT(A) \rightarrow [0, \infty)$  by

$$\iota(\langle a \rangle)(\tau) = d_\tau(\langle a \rangle), \forall \tau \in QT(A).$$

Clearly  $\iota(\langle a \rangle)$  is well defined with and  $\iota(\langle a \rangle) \in LAff_b(QT(A))^+$  for all  $\langle a \rangle \in W(A)$ . Note that  $\iota$  defines a morphism from  $(W(A), \leq, \langle 1_A \rangle)$  to  $(LAff_b(QT(A))^+, \leq, 1)$  where 1 is the constant function 1 on  $QT(A)$  which is an order unit for  $LAff_b(QT(A))^+$ .

We end this section by recalling the definition of radius of comparison of a  $C^*$ -algebra.

**Definition 2.12.** Let  $A$  be a  $C^*$ -algebra.  $A$  has finite radius of comparison if there is some real number  $r > 0$  such that the following hold for all  $a, b \in M_\infty(A)_+$ ;

$$(2.1) \quad (d_\tau(\langle a \rangle) + r \leq d_\tau(\langle b \rangle), \forall \tau \in QT(A)) \implies a \preceq b.$$

If  $A$  is of finite radius of comparison, the radius of comparison of  $A$  ( $rc(A)$ ) is the infimum of all  $r$  as in 2.1. If not the radius of comparison is infinite and we write  $rc(A) = \infty$ . Note that  $rc(A) = 0$  iff  $A$  has strict comparison.

### 3. PROOF OF THE MAIN RESULTS

Unless stated otherwise all  $C^*$ -algebras are assumed to be unital and stably finite. Recall that in the non stably finite case conjectures hold trivially.

**Theorem 3.1.** *Let  $A$  be a unital  $C^*$ -algebra. Then  $LDF(A)$  is dense in  $DF(A)$  if and only if  $\iota : (W(A), \leq, \langle 1_A \rangle) \rightarrow (LAff_b(QT(A))^+, \leq, 1)$  is a stable order embedding, where  $LAff_b(QT(A))^+$  and  $\iota$  is as defined in the previous section and 1 denote the constant function 1.*

*Proof.* Suppose  $\iota$  is a stable order embedding.

Let  $K$  denote the closure (in pointwise convergence) of  $LDF(A)$  in  $DF(A)$ . Then  $K$  is a compact convex subset of  $DF(A)$ . Suppose  $x, y \in W(A)$  are such that  $d(x) < d(y)$  for all  $d \in K$ . The function defined on  $K$  given by  $d \mapsto d(x) - d(y)$  is strictly positive and continuous on  $K$  in pointwise topology. Since  $K$  is compact the function attains a minimum  $\delta > 0$  on  $K$ .

Choose some  $n \in \mathbb{N}$  large enough so that  $n\delta > 1$ .

Then,

$$nd(x) + 1 \leq nd(y), \forall d \in K.$$

In particular,

$$nd_\tau(x) + 1 \leq nd_\tau(y), \forall \tau \in QT(A).$$

Therefore,

$$nu(x) + 1 \leq nu(y).$$

Hence, as  $\iota$  is a stable order embedding, there is some  $m \in \mathbb{N}$  and  $z \in W(A)$  such that,

$$mnx + z + \langle 1_A \rangle \leq mny + z.$$

Then it follows,

$$s(x) < s(y), \forall s \in DF(A).$$

Therefore, by Lemma 2.4  $K = DF(A)$ , i.e.  $LDF(A)$  is dense in  $DF(A)$ .

Now suppose  $LDF(A)$  is dense in  $DF(A)$ .

Note that in general  $\iota$  is an order preserving homomorphism. To verify its a stable order embedding let  $x, y \in W(A)$  and suppose that there is some  $n \in \mathbb{N}$  such that,

$$nu(x) + 1 \leq nu(y).$$

Then for all  $\tau \in QT(A)$ ,

$$d_\tau(nx + \langle 1_A \rangle) \leq d_\tau(ny).$$

Therefore, since  $LDF(A)$  is dense in  $DF(A)$  by assumption,

$$\begin{aligned} s(nx + \langle 1_A \rangle) &\leq s(ny), \forall s \in DF(A) \\ s(nx) &< s(ny), \forall s \in DF(A). \end{aligned}$$

Therefore by Lemma 2.3, there is some  $m \in \mathbb{N}$  and  $z \in W(A)$  such that,

$$mnx + \langle 1_A \rangle + z \leq mny + z$$

and  $\iota$  is a stable order embedding.  $\square$

Given  $s \in DF(A)$  define  $\overline{s} : W(A) \rightarrow [0, \infty)$  by

$$\overline{s}(\langle a \rangle) = \sup_{\epsilon > 0} s(\langle (a - \epsilon)_+ \rangle).$$

We need the following Proposition from [13] (c.f [3]) to prove part 2 of Theorem 3.4.

**Proposition 3.2.** [13, Proposition 4.1] *Let  $A$  be a  $C^*$ -algebra and let  $s \in DF(A)$ . Then  $\overline{s}$  defined above is a well defined lower semicontinuous dimension function and  $\overline{s}(\langle a \rangle) \leq s(\langle a \rangle)$  for all  $a \in M_\infty(A)_+$ .*

The following is mainly a consequence of Krein-Milman Theorem.

**Lemma 3.3.** *Suppose  $A$  is a unital  $C^*$ -algebra with  $\partial_e(QT(A))$  finite and non empty. Then  $LDF(A)$  is compact and moreover the map  $g : QT(A) \rightarrow LDF(A)$  given by  $\tau \mapsto d_\tau$  is an affine homeomorphism. If its also the case that Conjecture 1.1 holds for  $A$  then  $DF(A) = LDF(A)$  and  $DF(A)$  is affinely homeomorphic to  $QT(A)$ .*

*Proof.* From Theorem 2.11,  $g : QT(A) \rightarrow LDF(A)$  is an affine bijection and  $g^{-1}$  is continuous. Since  $QT(A)$  is compact and convex, by Krein-Milman Theorem  $QT(A)$  is the closure of the convex hull of  $\partial_e(QT(A))$ . As  $\partial_e(QT(A))$  is assumed to be finite, its convex hull  $co(\partial_e(QT(A)))$  is compact and therefore we in fact have;

$$QT(A) = \overline{co(\partial_e(QT(A)))} = co(\partial_e(QT(A))).$$

Thus,

$$LDF(A) = g(QT(A)) = g(co(\partial_e(QT(A))))$$

and on the other hand since  $g$  is an affine bijection  $g(\partial_e(QT(A))) = \partial_e(LDF(A))$ . Therefore,

$$LDF(A) = g(co(\partial_e(QT(A)))) = co(\partial_e(LDF(A))).$$

In particular, since  $\partial_e(LDF(A)) = g(co(\partial_e(QT(A))))$  is a non empty finite set,  $LDF(A)$  is compact and  $g$  is a homeomorphism.

Now if Conjecture 1.1 is true then  $DF(A) = \overline{LDF(A)}$ . As we had just noted,  $LDF(A)$  is compact and so it is equal to its own closure. Thus  $DF(A) = LDF(A)$  and is affinely homeomorphic to  $QT(A)$  from the preceding paragraph.  $\square$



**Theorem 3.4.** *Let  $A$  be any unital  $C^*$ -algebra. The following hold.*

1. *If  $A$  has finite radius of comparison then  $LDF(A)$  is dense in  $DF(A)$ .*
2. *If  $W(A)$  almost unperforated then  $LDF(A)$  is dense in  $DF(A)$ .*
3. *If  $\partial_e(QT(A))$  is a finite set and if either of the assumptions above (in 1,2) holds for  $A$  then  $DF(A) = LDF(A)$  and  $DF(A)$  is affinely homeomorphic to  $QT(A)$ . In particular  $DF(A)$  is a Choquet simplex.*

*Proof.* Proof of 1:

Let  $rc(A) = r < \infty$ . By Theorem 3.1 we only have to show that  $\iota$  is a stable order embedding.

Let  $x, y \in W(A)$  and suppose that there is some  $n \in \mathbb{N}$  such that,

$$n\iota(x) + 1 \leq n\iota(y).$$

Then for all  $\tau \in QT(A)$ ,

$$d_\tau(nx + \langle 1_A \rangle) \leq d_\tau(ny).$$

Choose some  $m \in \mathbb{N}$  large enough so that  $m > r + 1$ .

Then for all  $\tau \in QT(A)$ ,

$$\begin{aligned} d_\tau(mnx + \langle 1_A \rangle) + r &= d_\tau(mnx) + 1 + r \\ &< d_\tau(mnx) + m \\ &= md_\tau(nx + \langle 1_A \rangle) \\ &\leq md_\tau(ny) \end{aligned}$$

Therefore, since  $rc(A) = r$ ,

$$mnx + \langle 1_A \rangle \leq mny$$

and  $\iota$  is a stable order embedding.

Proof of 2:

Again we only have to show that  $\iota$  is a stable order embedding.

So let  $a, b \in M_\infty(A)_+$  and suppose that there is some  $n \in \mathbb{N}$  such that,

$$\iota(\langle a \rangle) + 1 \leq n\iota(\langle b \rangle)$$

Then for all  $\tau \in QT(A)$ ,

$$(3.1) \quad 2nd_\tau(\langle a \rangle) + 1 < 2nd_\tau(\langle b \rangle)$$

Fix  $\epsilon > 0$  and let  $s \in DF(A)$  be arbitrary. Then  $\bar{s} \in LDF(A)$ , where  $\bar{s}$  is as in Proposition 3.2.

Thus, by equation (3.1)

$$(3.2) \quad 2n\bar{s}(\langle a \rangle) + 1 < 2n\bar{s}(\langle b \rangle).$$

Note that by definition of  $\bar{s}$  we have,

$$s(\langle (a - \epsilon)_+ \rangle) \leq \bar{s}(\langle a \rangle).$$

Combining this with (3.2) we have,

$$\begin{aligned}
s(2n\langle(a - \epsilon)_+\rangle + 2\langle 1_A \rangle) &= 2ns(\langle(a - \epsilon)_+\rangle) + 2 \\
&\leq 2n\overline{s}(\langle a \rangle) + 1 + 1 \\
&< 2n\overline{s}(\langle b \rangle) + 1 \\
&\leq 2ns(\langle b \rangle) + 1 \\
&= s(2n\langle b \rangle + \langle 1_A \rangle)
\end{aligned}$$

Since  $2n\langle b \rangle + \langle 1_A \rangle$  is an order unit for  $W(A)$  and  $s$  is arbitrary, we apply Proposition 2.8 to conclude

$$2n\langle(a - \epsilon)_+\rangle + 2\langle 1_A \rangle \leq 2n\langle b \rangle + \langle 1_A \rangle.$$

Note that  $\epsilon$  is arbitrary and in particular does not depend on  $n$ .

Thus by [13, Proposition 2.4] it follows that,

$$2n\langle a \rangle + 2\langle 1_A \rangle \leq 2n\langle b \rangle + \langle 1_A \rangle.$$

In particular for  $z = \langle 1_A \rangle \in W(A)$ ,

$$2n\langle a \rangle + \langle 1_A \rangle + z \leq 2n\langle b \rangle + z$$

and we conclude that  $\iota : W(A) \rightarrow LAff_b(QT(A))^+$  is a stable order embedding. This complete the proof of 2.

Proof of 3: First part follows directly from Lemma 3.3 and parts 1 and 2 above. To see that  $DF(A)$  is a Choquet simplex recall  $QT(A)$  is Choquet.

□

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